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Long range variations on the Fibonacci universal code

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ABSTRACT

Fibonacci coding is based on Fibonacci numbers and was defined by Apostolico and Fraenkel (1987) [1]. Fibonacci numbers are generated by the recurrence relation $F_i = F_{i-1} + F_{i-2}$ $\forall i \geqslant 2$ with initial terms $F_0 = 1$, $F_1 = 1$. Variations on the Fibonacci coding are used in source coding as well as in cryptography. In this paper, we have extended the table given by Thomas [8]. We have found that there is no Gopala–Hemachandra code for a particular positive integer n and for a particular value of $a \in Z$. We conclude that for $n = 1, 2, 3, 4$, Gopala–Hemachandra code exists for $a = -2, -3, \ldots, -20$. Also, for $1 \leqslant n \leqslant 100$, there is at most m consecutive not available (N/A) Gopala–Hemachandra code in $GH_{-(4+m)}$ column where $1 \leqslant m \leqslant 16$. And, for $1 \leqslant n \leqslant 100$, as m increases the availability of Gopala–Hemachandra code decreases in $GH_{-(4+m)}$ column where $1 \leqslant m \leqslant 16$.

1. Introduction

Fibonacci coding is a universal code which encodes positive integers into binary codewords. All tokens end with 11 and have no 11 before the end. Every positive integer n has exactly one binary representation of the form $n = \sum_{i=1}^{p} a_i F_i$ where a_i is either 0 or 1 and F_i are the Fibonacci numbers $1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Let us define $F_0 = 1$ and $F_i = 0$ for $i < 0$. This representation has an interesting property, the string $a_1 a_2 a_3 \cdots$ does not contain any adjacent 1 bits. We use this property that the Fibonacci representation of an integer does not have any adjacent 1 bits. If n is a positive integer, we construct its Fibonacci representation and append a 1 bit to the result.

An another method to encode a positive integer n is

Step 1. Find the largest Fibonacci number equal to or less than n, keeping track of the remainder.

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Table 1

n	Zeckendorf's representation	Fibonacci representation	Fibonacci code, $F(n)$
1	F_1	1	11
2	F_2	01	011
3	F_3	001	0011
4	$F_1 + F_3$	101	1011
5	F_4	0001	00011
6	$F_1 + F_4$	1001	10011
7	$F_2 + F_4$	0101	01011
8	F_5	00001	000011
9	$F_1 + F_5$	10001	100011
10	$F_2 + F_5$	01001	010011
11	$F_3 + F_5$	00101	001011
12	$F_1 + F_3 + F_5$	10101	101011
13	F_6	000001	0000011
14	$F_1 + F_6$	100001	1000011
15	$F_2 + F_6$	010001	0100011

Step 2. If the number we subtracted was the ith unique Fibonacci number, put a one in the ith digit of our output.

Step 3. Repeat the previous steps, substituting our remainder for n, until we reach a remainder of zero.

Step 4. Place a one after the last naturally occurring one in our output.

To decode a token in the code, remove the last 1, assign the remaining bits the values $1, 2, 3, 5, 8, 13, 21, \ldots$ (the Fibonacci number) and add.

It is obvious that each of these codes ends with two adjacent 1 bits, so they can be decoded uniquely. A prefix code is a variable size code that satisfies the prefix attribute. The binary representation of the integers does not satisfy prefix attribute. One disadvantage of this representation is that the size n of the set of integers has to be known in advance since it determines the code size as $1 + \lfloor \log_2 n \rfloor$. The property of not having adjacent 1 bits restricts the number of binary patterns available for such codes, so they are longer than the other codes. In some applications, a prefix code is required to code a set of integers whose size is not known in advance.

Fibonacci code is of variable size code. Although, it is not asymptotically optimal, they perform well compared to the Elias code [3] as long as the number of source message is not too large. The Fibonacci code has the additional attribute of robustness, which manifests it self by the local containment of errors.

Zeckendorf's theorem states that every positive integer has a unique representation as the sum of nonconsecutive Fibonacci numbers [9]. An integer written in such a fashion is said to be in Zeckendorf's representation [2]. Therefore, while the recursive nature of the Fibonacci numbers allow some integers to have multiple representations using the above scheme. e.g. the decimal number 10 can be represented as $F_2 + F_3 + F_4$ or $F_2 + F_5$ but in Zeckendorf's representation, it will be $F_2 + F_5$. So Fibonacci representation of 10 is 01001 and Fibonacci code of 10 is 010011 by append a 1 bit to the Fibonacci representation. In this way, we can say that Zeckendorf's representation properly used in writing Fibonacci code for any positive integer n. The Fibonacci code for n is defined as $F(n) = a_1 a_2 a_3 \cdots a_p 1$. The Fibonacci code for small number is given in Table 1.

Fibonacci coding has a useful property that sometimes makes it attractive in comparison to other universal codes. It is easier to recover data from a damaged stream. With most other universal codes, if a single bit is altered, none of the data comes after it will be correctly read. On the other hand, with Fibonacci coding, a changed bit may cause one token to be read as two, or cause two tokens to be read incorrectly as one, but reading a 0 from the stream will stop the errors from propagating further. Since the only stream that has no 0 in it is a stream of 11 tokens, the total edit distance between a stream damaged by a single bit error and the original stream is at most three.

Table 2

	GH_{-2}	GH_{-3}	GH_{-4}	GH_{-5}	GH_{-6}	GH_{-7}	GH_{-8}	GH_{-9}	GH_{-10}	GH_{-11}
1	0011	0011	0011	0011	0011	0011	0011	0011	0011	0011
2	10011	10011	10011	10011	10011	10011	10011	10011	10011	10011
3	011	100011	100011	100011	1 00011	100011	100011	100011	100011	100011
4	00011	011	101011	101011	101011	101011	101011	101011	101011	101011
5	000011	00011	011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
6	001011	000011	00011	011	N/A	N/A	N/A	N/A	N/A	N/A
7	01011	001011	000011	00011	011	N/A	N/A	N/A	N/A	N/A
8	1010011	1000011	001011	000011	00011	011	N/A	N/A	N/A	N/A
9	0000011	1010011	1000011	001011	000011	00011	011	N/A	N/A	N/A
10	0010011	010011	1010011	1000011	001011	000011	00011	011	N/A	N/A
11	1001011	0000011	01011	1010011	1000011	001011	000011	00011	011	N/A
12	0100011	0010011	010011	N/A	1010011	1000011	001011	000011	00011	011
13	10100011	1001011	0000011	01011	N/A	1010011	1000011	001011	000011	00011
14	00000011	10000011	0010011	010011	N/A	N/A	1010011	1000011	001011	000011
15	00100011	10100011	1001011	0000011	01011	N/A	N/A	1010011	1000011	001011
16	10010011	0001011	10000011	0010011	010011	N/A	N/A	N/A	1010011	1000011
17	01000011	00000011	10100011	1001011	0000011	01011	N/A	N/A	N/A	1010011
18	10101011	00100011	0100011	1 0000011	0010011	010011	N/A	N/A	N/A	N/A
19	00001011	10010011	0001011	10100011	1001011	1000011	01011	N/A	N/A	N/A
20	00101011	0101011	00000011	N/A	10000011	0010011	010011	N/A	N/A	N/A
21	01 01 0011	10101011	001 00011	0100011	10100011	1001011	0000011	01011	N/A	N/A
22	101000011	00010011	10010011	0001011	N/A	10000011	0010011	010011	N/A	N/A
23	000000011	00001011	10001011	00000011	N/A	10100011	1001011	0000011	01011	N/A
24	001000011	00101011	0101011	00100011	0100011	N/A	10000011	0010011	010011	N/A
25	100100011	100000011	01 000011	10010011	0001011	N/A	10100011	1001011	0000011	01011
26	100010011	01010011	00010011	10001011	00000011	N/A	N/A	10000011	0010011	010011
27	101010011	01001011	00001011	10101011	00100011	0100011	N/A	10100011	1001011	0000011
28	000010011	000000011	00101011	0101011	10010011	0001011	N/A	N/A	10000011	0010011
29	001010011	001000011	100000011	01000011	10001011	00000011	N/A	N/A	10100011	1001011
30	010100011	100100011	101000011	00010011	10101011	00100011	0100011	N/A	N/A	10000011
31	101001011	100010011	01010011	00001011	N/A	10010011	0001011	N/A	N/A	10100011
32	000001011	101010011	01001011	00101011	0101011	10001011	00000011	N/A	N/A	N/A
33	001001011	000100011	000000011	100000011	01000011	10101011	00100011	0100011	N/A	N/A
34	100101011	000010011	001000011	101000011	00010011	N/A	10010011	0001011	N/A	N/A
35	010001011	001010011	100100011	N/A	00001011	N/A	10001011	00000011	N/A	N/A
36	1010000011	100001011	100010011	01010011	00101011	0101011	10101011	00100011	0100011	N/A
37	0000000011	101001011	101010011	01001011	100000011	01000011	N/A	10010011	0001011	N/A
38	0010000011	010010011	010000011	000000011	101000011	00010011	N/A	10001011	00000011	N/A
39	1001000011	000001011	000100011	001000011	N/A	00001011	N/A	10101011	00100011	0100011
40	1000100011	001001011	000010011	100100011	N/A	00101011	0101011	N/A	10010011	0001011
41	1010100011	100101011	001010011	100010011	01010011	100000011	01000011	N/A	10001011	00000011
42	0000100011	1000000011	100001011	101010011	01001011	101000011	00010011	N/A	10101011	00100011
43	0010100011	010001011	101001011	N/A	000000011	N/A	00001011	N/A	N/A	10010011
44	0101000011	000101011	010100011	010000011	001000011	N/A	00101011	0101011	N/A	10001011
45	1010010011	0000000011	010010011	000100011	100100011	N/A	100000011	01000011	N/A	10101011
46	0000010011	0010000011	000001011	000010011	100010011	01010011	101000011	00010011	N/A	N/A
47	0010010011	1001000011	001001011	001010011	101010011	01001011	N/A	00001011	N/A	N/A
48	1001010011	010101011	100101011	100001011	N/A	000000011	N/A	00101011	0101011	N/A
49	1000001011	1010100011	1000000011	101001011	N/A	001000011	N/A	100000011	01000011	N/A
50	1010001011	0001000011	1010000011	N/A	010000011	100100011	N/A	101000011	00010011	N/A

2. Gopala–Hemachandra (GH) sequence and codes

A variation to the Fibonacci sequence is the more general GH sequence [6] $\{a, b, a+b, a+2b, 2a+3b, 3a+5b, \ldots\}$ for any pair a, b which for the case $a = 1$, $b = 2$ represents the Fibonacci numbers [4,5,7] gives historical details of these sequences.

Now, we introduce a variation on the Fibonacci coding scheme by using the GH sequence. The second order Variant Fibonacci sequence, $VF_a(n)$ is defined as the GH sequence $\{a, b, a+b, a+2b, 2a+3b, 3a+5b, \ldots\}$ where $b = 1 - a$, That is, $VF_a(1) = a$ $(a \in Z)$; $VF_a(2) = 1 - a$; and for $n \geqslant 3$, $VF_a(n) = VF_a(n-1) + VF_a(n-2)$.

From the above definition for $a = -2$, we have $\{-2, 3, 1, 4, 5, 9, 14, 23, \ldots\}$.

Table 3

	GH_{-2}	GH_{-3}	GH_{-4}	GH_{-5}	GH_{-6}	GH_{-7}	GH_{-8}	GH_{-9}	GH_{-10}	GH_{-11}
51	0000001011	0000100011	010001011	010100011	000100011	100010011	01010011	N/A	00001011	N/A
52	0010001011	0010100011	000101011	010010011	000010011	101010011	01001011	N/A	00101011	0101011
53	0101010011	1000010011	0000000011	000001011	001010011	N/A	000000011	N/A	100000011	01000011
54	0100001011	1010010011	0010000011	001001011	100001011	N/A	001000011	N/A	101000011	00010011
55	0001001011	0100100011	1001000011	100101011	101001011	N/A	100100011	N/A	N/A	00001011
56	0000101011	0000010011	1000100011	1000000011	N/A	010000011	100010011	01010011	N/A	00101011
57	0010101011	0010010011	010101011	1010000011	N/A	000100011	101010011	01001011	N/A	100000011
58	10000000011	1001010011	0100000011	N/A	010100011	000010011	N/A	000000011	N/A	101000011
59	0100101011	1000001011	0001000011	010001011	010010011	001010011	N/A	001000011	N/A	N/A
60	00000000011	0100010011	0000100011	000101011	000001011	100001011	N/A	100100011	N/A	N/A
61	00100000011	0001010011	0010100011	0000000011	001001011	101001011	N/A	100010011	01010011	N/A
62	10010000011	0000001011	1000010011	0010000011	100101011	N/A	010000011	101010011	01001011	N/A
63	01000000011	0010001011	1010010011	1001000011	1000000011	N/A	000100011	N/A	000000011	N/A
64	10101000011	1001001011	0101000011	1000100011	1010000011	N/A	000010011	N/A	001000011	N/A
65	00001000011	1000101011	0100100011	1010100011	N/A	010100011	001010011	N/A	100100011	N/A
66	00101000011	1010101011	0000010011	010101011	N/A	010010011	100001011	N/A	100010011	01010011
67	10000100011	0001001011	0010010011	0100000011	010001011	000001011	101001011	N/A	101010011	01001011
68	10100100011	0000101011	1001010011	0001000011	000101011	001001011	N/A	010000011	N/A	000000011
69	00000100011	0010101011	1000001011	0000100011	0000000011	100101011	N/A	000100011	N/A	001000011
70	00100100011	10000000011	1010001011	0010100011	0010000011	1000000011	N/A	000010011	N/A	100100011
71	10010100011	0101001011	0100010011	1000010011	1001000011	1010000011	N/A	001010011	N/A	100010011
72	01000100011	0100101011	0001010011	1010010011	1000100011	N/A	010100011	100001011	N/A	101010011
73	10100010011	00000000011	0000001011	N/A	1010100011	N/A	010010011	101001011	N/A	N/A
74	00000010011	00100000011	0010001011	0101000011	N/A	N/A	000001011	N/A	010000011	N/A
75	001 00010011	10010000011	1001001011	0100100011	010101011	010001011	001001011	N/A	000100011	N/A
76	10010010011	10001000011	1000101011	0000010011	0100000011	000101011	100101011	N/A	000010011	N/A
77	01000010011	10101000011	0101010011	0010010011	0001000011	0000000011	1000000011	N/A	001010011	N/A
78	10101010011	00010000011	0100001011	1001010011	0000100011	0010000011	1010000011	N/A	100001011	N/A
79	00001010011	00001000011	0001001011	1000001011	0010100011	1001000011	N/A	010100011	101001011	N/A
80	00101010011	00101000011	0000101011	1010001011	1000010011	1000100011	N/A	010010011	N/A	010000011
81	01010010011	10000100011	0010101011	N/A	1010010011	1010100011	N/A	000001011	N/A	000100011
82	01001010011	10100100011	00000000011	0100010011	N/A	N/A	N/A	001001011	N/A	000010011
83	00000001011	01001000011	00100000011	0001010011	N/A	N/A	010001011	100101011	N/A	001010011
84	00100001011	00000100011	0101001011	0000001011	0101000011	010101011	000101011	1000000011	N/A	100001011
85	10010001011	00100100011	0100101011	0010001011	0100100011	0100000011	0000000011	1010000011	N/A	101001011
86	10001001011	10010100011	00000000011	1001001011	0000010011	0001000011	0010000011	N/A	010100011	N/A
87	10101001011	10000010011	00100000011	1000101011	0010010011	0000100011	1001000011	N/A	010010011	N/A
88	00001001011	10100010011	10010000011	1010101011	1001010011	0010100011	1000100011	N/A	000001011	N/A
89	00101001011	00010100011	10001000011	0101010011	1000001011	1000010011	1010100011	N/A	001001011	N/A
90	01010001011	00000010011	10101000011	0100001011	1010001011	1010010011	N/A	N/A	100101011	N/A
91	10100101011	00100010011	01000000011	0001001011	N/A	N/A	N/A	010001011	1000000011	N/A
92	00000101011	10010010011	00010000011	0000101011	N/A	N/A	N/A	000101011	1010000011	N/A
93	00100101011	01010100011	00001000011	0010101011	0100010011	N/A	010101011	0000000011	N/A	010100011
94	10010101011	10101010011	00101000011	10000000011	0001010011	0101000011	0100000011	0010000011	N/A	010010011
95	01000101011	00010010011	10000100011	10100000011	0000001011	0100100011	0001000011	1001000011	N/A	000001011
96	00010101011	00001010011	10100100011	N/A	0010001011	0000010011	0000100011	1000100011	N/A	001001011
97	000000000011	00101010011	01010000011	0101001011	1001001011	0010010011	0010100011	1010100011	N/A	100101011
98	001000000011	10000001011	01001000011	0100101011	1000101011	1001010011	1000010011	N/A	N/A	1000000011
99	01010101011	01010010011	00000100011	00000000011	1010101011	1000001011	1010010011	N/A	010001011	1010000011
100	100010000011	01001010011	00100100011	00100000011	N/A	1010001011	N/A	N/A	000101011	N/A

For different values of a, we find different sequences. Daykin proved that only the standard Fibonacci sequence gives a unique Zeckendorf's representation for all positive integers [2]. But Variant Fibonacci sequences allow for multipale Zeckendorf's representations of the same integer. James Harold Thomas showed that for a sequence $VF_{-5}(n) = \{-5, 6, 1, 7, 8, 15, 23, 38, \ldots\}$ there is no Zeckendorf's representation for integer $n = 5, 12$ [8]. In this paper, we study for what values of n the Zeckendorf's representation does not available for $a \leqslant -2$. For $n = 1, 2, 3, \ldots, 100$ the representation or not available (N/A) of GH codes for $a \leqslant -2$ are displayed in Tables 2–5.

From the above Tables 2–5, we conclude that

1. For $n = 1, 2, 3, 4,$ GH code exists for $a = -2, -3, \ldots, -20$.
2. For $1 \leqslant n \leqslant 100$, there is at most m consecutive not available (N/A) GH code in $GH_{-(4+m)}$ column where $1 \leqslant m \leqslant 16$.
3. For $1 \leqslant n \leqslant 100$, as m increases the availability of GH code decreases in $GH_{-(4+m)}$ column where $1 \leqslant m \leqslant 16$.

   GH codes are longer than standard Fibonacci code, so they are less desirable. The family of GH codes $GH_a(n)$, which satisfy our condition for encoding all of the desirable integers

Table 4

	GH_{-12}	GH_{-13}	GH_{-14}	GH_{-15}	GH_{-16}	GH_{-17}	GH_{-18}	GH_{-19}	GH_{-20}
1	0011	0011	0011	0011	0011	0011	0011	0011	0011
2	10011	10011	10011	10011	10011	10011	10011	10011	10011
3	100011	100011	100011	100011	100011	100011	100011	100011	100011
4	101011	101011	101011	101011	101011	101011	101011	101011	101011
5	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
6	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
7	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
8	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
9	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
10	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
11	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
12	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
13	011	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
14	00011	011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
15	000011	00011	011	N/A	N/A	N/A	N/A	N/A	N/A
16	001011	000011	00011	011	N/A	N/A	N/A	N/A	N/A
17	1000011	001011	000011	00011	011	N/A	N/A	N/A	N/A
18	1010011	1000011	001011	000011	00011	011	N/A	N/A	N/A
19	N/A	1010011	1000011	001011	000011	00011	011	N/A	N/A
20	N/A	N/A	1010011	1000011	001011	000011	00011	011	N/A
21	N/A	N/A	N/A	1010011	1000011	001011	000011	00011	011
22	N/A	N/A	N/A	N/A	1010011	1000011	001011	000011	00011
23	N/A	N/A	N/A	N/A	N/A	1010011	1000011	001011	000011
24	N/A	N/A	N/A	N/A	N/A	N/A	1010011	1000011	001011
25	N/A	N/A	N/A	N/A	N/A	N/A	N/A	1010011	1000011
26	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	1010011
27	01011	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
28	010011	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
29	0000011	01011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
30	0010011	010011	N/A	N/A	N/A	N/A	N/A	N/A	N/A
31	1001011	0000011	01011	N/A	N/A	N/A	N/A	N/A	N/A
32	10000011	0010011	010011	N/A	N/A	N/A	N/A	N/A	N/A
33	10100011	1001011	0000011	01011	N/A	N/A	N/A	N/A	N/A
34	N/A	10000011	0010011	010011	N/A	N/A	N/A	N/A	N/A
35	N/A	10100011	1001011	0000011	01011	N/A	N/A	N/A	N/A
36	N/A	N/A	10000011	0010011	010011	N/A	N/A	N/A	N/A
37	N/A	N/A	10100011	1001011	0000011	01011	N/A	N/A	N/A
38	N/A	N/A	N/A	10000011	0010011	010011	N/A	N/A	N/A
39	N/A	N/A	N/A	10100011	1001011	0000011	01011	N/A	N/A
40	N/A	N/A	N/A	N/A	10000011	0010011	010011	N/A	N/A
41	N/A	N/A	N/A	N/A	10100011	1001011	0000011	01011	N/A
42	0100011	N/A	N/A	N/A	N/A	10000011	0010011	010011	N/A
43	0001011	N/A	N/A	N/A	N/A	10100011	1001011	0000011	01011
44	00000011	N/A	N/A	N/A	N/A	N/A	10000011	0010011	010011
45	00100011	0100011	N/A	N/A	N/A	N/A	10100011	1001011	0000011
46	10010011	0001011	N/A	N/A	N/A	N/A	N/A	10000011	0010011
47	10001011	00000011	N/A	N/A	N/A	N/A	N/A	10100011	1001011
48	10101011	00100011	0100011	N/A	N/A	N/A	N/A	N/A	10000011
49	N/A	10010011	0001011	N/A	N/A	N/A	N/A	N/A	10100011
50	N/A	10001011	00000011	N/A	N/A	N/A	N/A	N/A	N/A

$1 \leqslant n \leqslant M$ (where M is the number of possible source message), allows us to have many more universal codes at our disposal when transmitting a message. Although, GH code e.g. $GH_{-5}(n), GH_{-6}(n), GH_{-7}(n), GH_{-8}(n), GH_{-9}(n), GH_{-10}(n), \ldots, GH_{-20}(n)$ which lack the ability to encode for certain values of n, can be used on portion of a message signal that contains only those source messages which they are to encode.

3. Conclusion

In this paper, encoding scheme is highly cryptographic nature. For $n = 1, 2, 3, 4,$ GH code exists for $a = -2, -3, \ldots, -20$. Also, for $1 \leqslant n \leqslant 100$, there is at most m consecutive not available (N/A) GH code in $GH_{-(4+m)}$ column where $1 \leqslant m \leqslant 16$. And, for $1 \leqslant n \leqslant 100$, as m increases the availabil-

Table 5

	GH_{-12}	GH_{-13}	GH_{-14}	GH_{-15}	GH_{-16}	GH_{-17}	GH_{-18}	GH_{-19}	GH_{-20}
51	N/A	10101011	00100011	0100011	N/A	N/A	N/A	N/A	N/A
52	N/A	N/A	10010011	0001011	N/A	N/A	N/A	N/A	N/A
53	N/A	N/A	10001011	00000011	N/A	N/A	N/A	N/A	N/A
54	N/A	N/A	10101011	00100011	0100011	N/A	N/A	N/A	N/A
55	N/A	N/A	N/A	10010011	0001011	N/A	N/A	N/A	N/A
56	0101011	N/A	N/A	10001011	00000011	N/A	N/A	N/A	N/A
57	01000011	N/A	N/A	10101011	00100011	0100011	N/A	N/A	N/A
58	00010011	N/A	N/A	N/A	10010011	0001011	N/A	N/A	N/A
59	00001011	N/A	N/A	N/A	10001011	00000011	N/A	N/A	N/A
60	00101011	0101011	N/A	N/A	10101011	00100011	0100011	N/A	N/A
61	100000011	01000011	N/A	N/A	N/A	10010011	0001011	N/A	N/A
62	101000011	00010011	N/A	N/A	N/A	10001011	00000011	N/A	N/A
63	N/A	00001011	N/A	N/A	N/A	10101011	00100011	0100011	N/A
64	N/A	00101011	0101011	N/A	N/A	N/A	10010011	0001011	N/A
65	N/A	100000011	01000011	N/A	N/A	N/A	10001011	00000011	N/A
66	N/A	101000011	00010011	N/A	N/A	N/A	10101011	00100011	0100011
67	N/A	N/A	00001011	N/A	N/A	N/A	N/A	10010011	0001011
68	N/A	N/A	00101011	0101011	N/A	N/A	N/A	10001011	00000011
69	N/A	N/A	100000011	01000011	N/A	N/A	N/A	10101011	00100011
70	N/A	N/A	101000011	00010011	N/A	N/A	N/A	N/A	10010011
71	01010011	N/A	N/A	00001011	N/A	N/A	N/A	N/A	10001011
72	01001011	N/A	N/A	00101011	0101011	N/A	N/A	N/A	10101011
73	000000011	N/A	N/A	100000011	01000011	N/A	N/A	N/A	N/A
74	001000011	N/A	N/A	101000011	00010011	N/A	N/A	N/A	N/A
75	100100011	N/A	N/A	N/A	00001011	N/A	N/A	N/A	N/A
76	100010011	01010011	N/A	N/A	00101011	0101011	N/A	N/A	N/A
77	101010011	01001011	N/A	N/A	100000011	01000011	N/A	N/A	N/A
78	N/A	000000011	N/A	N/A	101000011	00010011	N/A	N/A	N/A
79	N/A	001000011	N/A	N/A	N/A	00001011	N/A	N/A	N/A
80	N/A	100100011	N/A	N/A	N/A	00101011	0101011	N/A	N/A
81	N/A	100010011	01010011	N/A	N/A	100000011	01000011	N/A	N/A
82	N/A	101010011	01001011	N/A	N/A	101000011	00010011	N/A	N/A
83	N/A	N/A	000000011	N/A	N/A	N/A	00001011	N/A	N/A
84	N/A	N/A	001000011	N/A	N/A	N/A	00101011	0101011	N/A
85	N/A	N/A	100100011	N/A	N/A	N/A	100000011	01000011	N/A
86	010000011	N/A	100010011	01010011	N/A	N/A	101000011	00010011	N/A
87	000100011	N/A	101010011	01001011	N/A	N/A	N/A	00001011	N/A
88	000010011	N/A	N/A	000000011	N/A	N/A	N/A	00101011	0101011
89	001010011	N/A	N/A	001000011	N/A	N/A	N/A	100000011	01000011
90	100001011	N/A	N/A	100100011	N/A	N/A	N/A	101000011	00010011
91	101001011	N/A	N/A	100010011	01010011	N/A	N/A	N/A	00001011
92	N/A	010000011	N/A	101010011	01001011	N/A	N/A	N/A	00101011
93	N/A	000100011	N/A	N/A	000000011	N/A	N/A	N/A	100000011
94	N/A	000010011	N/A	N/A	001000011	N/A	N/A	N/A	101000011
95	N/A	001010011	N/A	N/A	100100011	N/A	N/A	N/A	N/A
96	N/A	100001011	N/A	N/A	100010011	01010011	N/A	N/A	N/A
97	N/A	101001011	N/A	N/A	101010011	01001011	N/A	N/A	N/A
98	N/A	N/A	010000011	N/A	N/A	000000011	N/A	N/A	N/A
99	N/A	N/A	000100011	N/A	N/A	001000011	N/A	N/A	N/A
100	010100011	N/A	000010011	N/A	N/A	100100011	N/A	N/A	N/A

ity of GH code decreases in $GH_{-(4+m)}$ column where $1 \leqslant m \leqslant 16$. In this paper, we display tables $GH_{-a}(n)$ for $2 \leqslant a \leqslant 20$ and $1 \leqslant n \leqslant 100$. Since the GH codes are determined by their initial value a, the codebook could be easily changed multiple times during transmission, making decoding much more difficult. The presence of multiple representation of the same integer allow for a codebook that appears larger than it actually is. The codeword lengths are not always increasing so it gives cryptographic advantages.

If any researcher is interested in carrying out further research and wants to know the behavior of $GH_{-a}(n)$ for any value of $a \geqslant 2$ and n, we can provide the result.

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